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SCATTERING OF ANTI-PLANE SHEAR WAVES IN A FUNCTIONALLY GRADED MATERIAL STRIP WITH AN OFF-CENTER VERTICAL CRACK *

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Abstract: The scattering problem of anti-plane shear waves in a functionally graded material strip with an off-center crack is investigated by use of Schmidt method. The crack is vertically to the edge of the strip. By using the Fourier transform, the problem can be solved with the help of a pair of dual integral equations that the unknown variable is the jump of the displacement across the crack surfaces. To solve the dual integral equations, the jump of the displacement across the crack surfaces was expanded in a series of Jacobi polynomials. Numerical examples were provided to show the effects of the parameter describing the functionally graded materials, the position of the crack and the frequency of the incident waves upon the stress intensity factors of the crack.

Key words: crack; elastic waves; functionally graded materials

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Introduction

In recent years, functionally graded materials (FGMs) have been widely introduced and applied to the development of thermal and structural components due to its ability not only to reduce the residual and thermal stresses but also to increase the bonding strength and toughness as well. In an attempt to address the issues pertaining to the fracture analysis of bonded media with such transitional interfacial properties, a series of solutions to certain crack problems was obtained by Erdogan and his associates[1,2]. Among them there are some solutions for a FGM strip containing an imbedded or an edge crack perpendicular to the surfaces[1]; some for a crack in the non-homogeneous interlayer bounded by dissimilar homogeneous media[2]. Similar problems of delamination or an interface crack between a functionally graded coating and a substrate were considered in Refs.[3,4]. The crack problem in FGM layers under thermal stresses was studied by Erdogan and Wu[5]. The steady state dynamic fracture of FGMs, under in-plane loading with the material properties being assumed to vary along the direction of crack propagation, was reported in Ref.[6]. The dynamic crack problem for non-homogeneous composite materials was considered in Ref.[7] but they considered the FGM layer as a multi-layered homogeneous medium. Experimental studies of the dynamic fracture of FGMs with discrete property variation using photoelasticity technique were also conducted in Ref.[8]. In spite of these efforts, the understanding of the dynamic fracture process of FGMs is still limited

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due to the mathematical complexities. To our knowledge, the dynamic behavior of FGMs with an off-center vertical crack subjected to the harmonic anti-plane shear stress waves has not been studied. Thus, the present work is an attempt to fill this information needed.

In this study, the scattering problem of anti-plane shear waves in a functionally graded material strip with an off-center vertical crack is investigated by use of Schmidt method[9]. The crack is vertically to the edge of the strip. To make the analysis tractable, it is assumed that the shear modulus and the density vary exponentially with coordinate parallel to the crack. Fourier transform is applied and a mixed boundary value problem is reduced to a pair of dual integral equations. To solve the dual integral equations, the jump of the displacement across the crack surfaces is expanded in a series of Jacobi polynomials. This process is different from that adopted in previous works[1−8]. Numerical examples are provided to show the effects of the parameter describing the functionally graded materials, the position of the crack and the frequency of the incident waves upon the stress intensity factors of the crack.

1 Formulation of Problem

Consider an infinite long, thin functionally graded material strip of width $2h\,(2h > b)$, containing an off-center crack of length $2l = b-a$ along the x-axis. The crack is vertically to the edge of the strip as shown in Fig.1. The harmonic anti-plane shear waves are vertically incident. Let ω be the circular frequency of the incident wave. $w_0^{(j)}(x,y,t)(j=1,2)$is the mechanical displacement. $\tau_{zk0}^{(j)}(x,y,t)(k=x,y,\ j=1,2)$ is the anti-plane shear stress. Also note that all quantities with superscript $j(j=1,2)$ refer to the upper half plane 1 $(y\geq 0)$and the lower half plane $2(y\leq 0)$ as shown in Fig.1, respectively. Because the incident waves are the harmonic anti-plane shear stress waves, all field quantities of $w_0^{(j)}(x,y,t)$ and $\tau_{zk0}^{(j)}(x,y,t)$ can be assumed to be of the forms as follows:

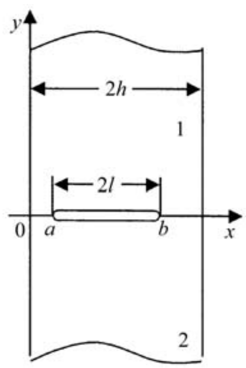

Fig.1 Geometry of off-center crack problem in a functionally graded material strip

$$[w_0^{(j)}(x,y,t),\tau_{zk0}^{(j)}(x,y,t)] = [w^{(j)}(x,y),\tau_{zk}^{(j)}(x,y)]\mathrm{e}^{-\mathrm{i}\omega t}. \tag{1}$$

In what follows, the time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$ will be suppressed but understood. As discussed in Ref.[10], the standard superposition technique was used in the present paper. So the boundary conditions of the present problem are

$$\tau_{yz}^{(1)}(x,0) = \tau_{yz}^{(2)}(x,0) = -\tau_0(x), \quad 0 < a \leq x \leq b < 2h; \tag{2}$$

$$\begin{cases} \tau_{yz}^{(1)}(x,0) = \tau_{yz}^{(2)}(x,0), \\ w^{(1)}(x,0) = w^{(2)}(x,0), \end{cases} \quad 0 \leq x \leq a, \quad b \leq x \leq 2h; \tag{3}$$

$$\tau_{xz}^{(j)}(0,y) = \tau_{xz}^{(j)}(2h,y) = 0, \quad |y| < \infty, \tag{4}$$

$$w^{(j)}(x,y) = 0, \quad \text{for } |y| \to \infty, \tag{5}$$

where $\tau_0(x)$ is a magnitude of the incident wave.

Crack problems in the non-homogeneous materials do not appear to be analytically tractable for arbitrary variations of material properties. Usually, one tries to generate the forms of non-homogeneities for which the problem becomes tractable. Similar to the treatment of the crack problem for isotropic non-homogeneous materials in Refs.[1–2], we assume that the material properties are described by

$$\mu = \mu_0 \mathrm{e}^{\beta x}, \quad \rho = \rho_0 \mathrm{e}^{\beta x}, \tag{6}$$

where μ is the shear modulus, ρ is the mass density of material. μ_0 and ρ_0 are two constants.

The constitutive equations for the mode III crack can be expressed as

$$\tau_{xz}^{(j)} = \mu\frac{\partial w^{(j)}}{\partial x}, \quad \tau_{yz}^{(j)} = \mu\frac{\partial w^{(j)}}{\partial y} \qquad (j=1,2). \tag{7}$$

The anti-plane governing equations for functionally graded materials are

$$\frac{\partial^2 w^{(j)}}{\partial y^2} + \beta\frac{\partial w^{(j)}}{\partial x} + \frac{\partial^2 w^{(j)}}{\partial x^2} = \rho\frac{\partial^2 w^{(j)}}{\partial t^2} \qquad (j=1,2). \tag{8}$$

2 Analysis

As discussed in Refs.[10,11], the solution of Eq.(8) can be assumed as

$$w^{(1)}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_1(s)\mathrm{e}^{-\gamma y}\mathrm{e}^{\mathrm{i}sx}ds + \frac{1}{\pi}\int_0^{\infty}[A_2(s)\mathrm{e}^{-\gamma x} + A_3(s)\mathrm{e}^{\gamma x}]\sin(sy)ds, \tag{9}$$

$$w^{(2)}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B_1(s)\mathrm{e}^{\gamma y}\mathrm{e}^{\mathrm{i}sx}ds + \frac{1}{\pi}\int_0^{\infty}[B_2(s)\mathrm{e}^{-\gamma x} + B_3(s)\mathrm{e}^{\gamma x}]\sin(sy)ds, \tag{10}$$

where $A_i(s)$ and $B_i(s)(i=1,2,3)$ are unknown functions. $\gamma = \sqrt{-\mathrm{i}s\beta + s^2 - \omega^2/c^2}$, $c = \sqrt{\mu_0/\rho_0}$, c is the shear wave velocities of the functionally graded materials. When $\beta = 0$, it will return to the homogenous material case.

So from Eq.(7), we have

$$\tau_{yz}^{(1)}(x,y) = \frac{-\mu_0\mathrm{e}^{\beta x}}{2\pi}\int_{-\infty}^{\infty}\gamma A_1(s)\mathrm{e}^{-\gamma y}\mathrm{e}^{\mathrm{i}sx}ds + \frac{2\mu_0\mathrm{e}^{\beta x}}{\pi}\int_0^{\infty}s[A_2(s)\mathrm{e}^{-\gamma x} + A_3(s)\mathrm{e}^{\gamma x}]\cos(sy)ds, \tag{11}$$

$$\tau_{xz}^{(1)}(x,y) = \frac{\mu_0\mathrm{e}^{\beta x}}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}sA_1(s)\mathrm{e}^{-\gamma y}\mathrm{e}^{\mathrm{i}sx}ds - \frac{2\mu_0\mathrm{e}^{\beta x}}{\pi}\int_0^{\infty}\gamma[A_2(s)\mathrm{e}^{-\gamma x} - A_3(s)\mathrm{e}^{\gamma x}]\sin(sy)ds, \tag{12}$$

$$\tau_{yz}^{(2)}(x,y) = \frac{\mu_0\mathrm{e}^{\beta x}}{2\pi}\int_{-\infty}^{\infty}\gamma B_1(s)\mathrm{e}^{\gamma y}\mathrm{e}^{\mathrm{i}sx}ds + \frac{2\mu_0\mathrm{e}^{\beta x}}{\pi}\int_0^{\infty}s[B_2(s)\mathrm{e}^{-\gamma x} + B_3(s)\mathrm{e}^{\gamma x}]\cos(sy)ds, \tag{13}$$

$$\tau_{xz}^{(2)}(x,y) = \frac{\mu_0\mathrm{e}^{\beta x}}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}sB_1(s)\mathrm{e}^{\gamma y}\mathrm{e}^{\mathrm{i}sx}ds - \frac{2\mu_0\mathrm{e}^{\beta x}}{\pi}\int_0^{\infty}\gamma[B_2(s)\mathrm{e}^{-\gamma x} - B_3(s)\mathrm{e}^{\gamma x}]\sin(sy)ds. \tag{14}$$

Substituting Eqs.(12) and (14) into Eq.(4), it can be obtained

$$\tau_{xz}^{(1)}(0,y) = \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}sA_1(s)\mathrm{e}^{-\gamma y}ds - \frac{2\mu_0}{\pi}\int_0^{\infty}\gamma[A_2(s) - A_3(s)]\sin(sy)ds = 0, \tag{15}$$

$$\begin{aligned}\tau_{xz}^{(1)}(2h,y) = {} & \frac{\mu_0\mathrm{e}^{2\beta h}}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}sA_1(s)\mathrm{e}^{-\gamma y}\mathrm{e}^{2\mathrm{i}sh}ds \\ & - \frac{2\mu_0\mathrm{e}^{2\beta h}}{\pi}\int_0^{\infty}\gamma[A_2(s)\mathrm{e}^{-2\gamma h} - A_3(s)\mathrm{e}^{2\gamma h}]\sin(sy)ds \\ = {} & 0,\end{aligned} \tag{16}$$

$$\tau_{xz}^{(2)}(0,y) = \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}sB_1(s)\mathrm{e}^{\gamma y}ds - \frac{2\mu_0}{\pi}\int_0^{\infty}\gamma[B_2(s) - B_3(s)]\sin(sy)ds = 0, \tag{17}$$

$$\begin{aligned}\tau_{xz}^{(2)}(2h,y)=&\frac{\mu_0 \mathrm{e}^{2\beta h}}{2\pi}\int_{-\infty}^{\infty}\mathrm{i}sB_1(s)\mathrm{e}^{\gamma y}\mathrm{e}^{2\mathrm{i}sh}ds\\&-\frac{2\mu_0\mathrm{e}^{2\beta h}}{\pi}\int_0^{\infty}\gamma[B_2(s)\mathrm{e}^{-2\gamma h}-B_3(s)\mathrm{e}^{2\gamma h}]\sin(sy)ds\\=&0.\end{aligned}\tag{18}$$

From Eqs.(15)–(18), it can be obtained that

$$A_2(t)-A_3(t)=\frac{\mathrm{i}t}{2\pi\gamma}\int_{-\infty}^{\infty}sA_1(s)\frac{1}{t^2+\gamma^2(s)}ds=d_1(t),\tag{19}$$

$$A_2(t)\mathrm{e}^{-2\gamma h}-A_3(t)\mathrm{e}^{2\gamma h}=\frac{\mathrm{i}t}{2\pi\gamma}\int_{-\infty}^{\infty}sA_1(s)\frac{\mathrm{e}^{2\mathrm{i}sh}}{t^2+\gamma^2(s)}ds=d_2(t),\tag{20}$$

$$B_2(t)-B_3(t)=-\frac{\mathrm{i}t}{2\pi\gamma}\int_{-\infty}^{\infty}sB_1(s)\frac{1}{t^2+\gamma^2(s)}ds=-g_1(t),\tag{21}$$

$$B_2(t)\mathrm{e}^{-2\gamma h}-B_3(t)\mathrm{e}^{2\gamma h}=-\frac{\mathrm{i}t}{2\pi\gamma}\int_{-\infty}^{\infty}sB_1(s)\frac{\mathrm{e}^{2\mathrm{i}sh}}{t^2+\gamma^2(s)}ds=-g_2(t).\tag{22}$$

Hence, it can be also obtained that

$$A_2(t)=\frac{d_1(t)\mathrm{e}^{2\gamma h}-d_2(t)}{\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h}},\quad A_3(t)=\frac{d_1(t)\mathrm{e}^{-2\gamma h}-d_2(t)}{\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h}},\tag{23}$$

$$B_2(t)=\frac{g_2(t)-g_1(t)\mathrm{e}^{2\gamma h}}{\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h}},\quad B_3(t)=\frac{g_2(t)-g_1(t)\mathrm{e}^{-2\gamma h}}{\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h}}.\tag{24}$$

From the boundary (3), Eqs.(11) and (13), it can be obtained that

$$\begin{aligned}&\frac{-1}{2\pi}\int_{-\infty}^{\infty}\gamma[A_1(s)+B_1(s)]\mathrm{e}^{\mathrm{i}sx}ds\\&+\frac{2}{\pi}\int_0^{\infty}s[A_2(s)\mathrm{e}^{-\gamma x}+A_3(s)\mathrm{e}^{\gamma x}-B_2(s)\mathrm{e}^{-\gamma x}-B_3(s)\mathrm{e}^{\gamma x}]ds=0.\end{aligned}\tag{25}$$

So from Eqs.(19)–(25), we have

$$A_1(s)=-B_1(s),\quad A_2(s)=B_2(s),\quad A_3(s)=B_3(s).\tag{26}$$

To solve the problem, the jump of displacements across the crack surfaces is defined as

$$f(x)=w^{(1)}(x,0)-w^{(2)}(x,0).\tag{27}$$

Substituting Eqs.(9)–(10) into Eq.(27), and applying Fourier transform technique, it can be obtained that

$$\bar{f}(s)=A_1(s)-B_1(s).\tag{28}$$

A superposed bar indicates the Fourier transform.

Substituting Eqs.(26) and (28) into Eq.(11), applying the boundary conditions (2) and (3), it can be obtained that

$$\begin{aligned}&\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\gamma\bar{f}(s)\mathrm{e}^{\mathrm{i}sx}ds\\&-\frac{i\mu_0}{2\pi^2}\int_0^{\infty}\frac{s^2\mathrm{e}^{-\gamma x}}{\gamma(s)(\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h})}[\mathrm{e}^{2\gamma h}\int_{-\infty}^{\infty}\bar{f}(u)\frac{u}{s^2+\gamma^2(u)}du-\int_{-\infty}^{\infty}\bar{f}(u)\frac{u\mathrm{e}^{2\mathrm{i}uh}}{s^2+\gamma^2(u)}du]ds\end{aligned}$$

$$-\frac{\mathrm{i}\mu_0}{2\pi^2}\int_0^\infty \frac{s^2\mathrm{e}^{\gamma x}}{\gamma(s)(\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h})}[\mathrm{e}^{-2\gamma h}\int_{-\infty}^\infty \bar{f}(u)\frac{u}{s^2+\gamma^2(u)}du-\int_{-\infty}^\infty \bar{f}(u)\frac{u\mathrm{e}^{2iuh}}{s^2+\gamma^2(u)}du]ds$$
$$=\tau_0(x)\mathrm{e}^{-\beta x},$$
$$0<a\le x\le b<2h, \qquad (29)$$

$$\int_{-\infty}^\infty \bar{f}(s)\mathrm{e}^{\mathrm{i}sx}ds=0, \qquad 0\le x<a, \quad b<x\le 2h. \qquad (30)$$

To determine the unknown functions $\bar{f}(s)$, the dual integral equations (29)–(30) must be solved.

3 Solution of Dual Integral Equations

To solve the present problem, the jump of the displacements across the crack surfaces can be represented by the following series:

$$f(x)=\sum_{n=0}^\infty a_n P_n^{(\frac{1}{2},\frac{1}{2})}(\frac{x-\frac{b+a}{2}}{\frac{b-a}{2}})(1-\frac{(x-\frac{b+a}{2})^2}{(\frac{b-a}{2})^2})^{\frac{1}{2}}, \qquad \text{for} \quad 0<a\le x\le b<2h, \qquad (31)$$

$$f(x)=0, \qquad \text{for} \quad 0\le x<a, \quad b<x\le 2h, \quad y=0, \qquad (32)$$

where a_n are unknown coefficients to be determined and $P_n^{(\frac{1}{2},\frac{1}{2})}(x)$ is a Jacobi polynomial[12]. The Fourier transformations[13] of Eqs.(31)–(32) are

$$\begin{cases}\bar{f}(s)=\sum_{n=0}^\infty a_n Q_n G(s)\frac{1}{s}\mathrm{J}_{n+1}(s\frac{b-a}{2}),\\ Q_n=2\sqrt{\pi}(-1)^n(i)^n\frac{\Gamma(n+1+\frac{1}{2})}{n!},\\ G(s)=\mathrm{e}^{-\mathrm{i}s\frac{a+b}{2}},\end{cases} \qquad (33)$$

where $\Gamma(x)$ and $\mathrm{J}_n(x)$ are the Gamma and Bessel functions, respectively. Substituting Eq.(33) into Eqs.(29)–(30), respectively, Eq.(30) can be automatically satisfied. Then, after integration with respect to x in $[a,x]$, the remaining Eq.(29) reduces to

$$-\frac{\mu_0}{4\pi}\sum_{n=0}^\infty a_nQ_n\int_{-\infty}^\infty \frac{i\gamma}{s^2}G(s)\mathrm{J}_{n+1}\left(s\frac{b-a}{2}\right)(\mathrm{e}^{\mathrm{i}sx}-\mathrm{e}^{\mathrm{i}sa})ds$$
$$=\int_a^x \tau_0(u)\mathrm{e}^{-\beta u}du-\frac{i\mu_0}{2\pi^2}\sum_{n=0}^\infty a_nQ_n\int_0^\infty \frac{s^2[\mathrm{e}^{-\gamma x}-\mathrm{e}^{-\gamma a}]}{\gamma^2(s)(\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h})}\cdot$$
$$[\mathrm{e}^{2\gamma h}\int_{-\infty}^\infty G(u)\mathrm{J}_{n+1}\left(u\frac{b-a}{2}\right)\frac{1}{s^2+\gamma^2(u)}du-\int_{-\infty}^\infty G(u)\mathrm{J}_{n+1}\left(u\frac{b-a}{2}\right)\frac{\mathrm{e}^{2iuh}}{s^2+\gamma^2(u)}du]ds$$
$$+\frac{i\mu_0}{2\pi^2}\sum_{n=0}^\infty a_nQ_n\int_0^\infty \frac{s^2[\mathrm{e}^{\gamma x}-\mathrm{e}^{\gamma a}]}{\gamma^2(s)(\mathrm{e}^{2\gamma h}-\mathrm{e}^{-2\gamma h})}\cdot$$
$$[\mathrm{e}^{-2\gamma h}\int_{-\infty}^\infty G(u)\mathrm{J}_{n+1}\left(u\frac{b-a}{2}\right)\frac{1}{s^2+\gamma^2(u)}du-\int_{-\infty}^\infty G(u)\mathrm{J}_{n+1}\left(u\frac{b-a}{2}\right)\frac{\mathrm{e}^{2iuh}}{s^2+\gamma^2(u)}du]ds,$$
$$a\le x\le b. \qquad (34)$$

For a large s, the integrands of the double semi-infinite integral in Eq.(34) almost have exponential forms. Thus the double semi-infinite integral in the Eq.(34) can be evaluated numerically. So it suffices to solve Eq.(34) for the present problem by the Schmidt method[9,14−16]. These can be seen in Refs.[14–16]. Here, they are omitted.

4 Numerical Calculation of Intensity Factors and Discussion

The coefficients a_n are known, so that the entire stress field. However, in fracture mechanics, it is of importance to determine the stress τ_{yz} in the vicinity of the crack tip. In the case of the present study, $\tau_{yz}^{(1)}$ along the crack line can be expressed as

$$\begin{aligned}
&\tau_{yz}^{(1)}(x,0)\\
=&-\frac{\mu_0 e^{\beta x}}{4\pi}\sum_{n=0}^{\infty}a_nQ_n\int_{-\infty}^{\infty}\frac{\gamma}{s}G(s)J_{n+1}\left(s\frac{b-a}{2}\right)e^{isx}ds\\
&+\frac{i\mu_0 e^{\beta x}}{2\pi^2}\sum_{n=0}^{\infty}a_nQ_n\int_0^{\infty}\frac{s^2e^{-\gamma x}}{\gamma(s)(e^{2\gamma h}-e^{-2\gamma h})}[e^{2\gamma h}\int_{-\infty}^{\infty}G(u)J_{n+1}\left(u\frac{b-a}{2}\right)\frac{1}{s^2+\gamma^2(u)}du\\
&-\int_{-\infty}^{\infty}G(u)J_{n+1}\left(u\frac{b-a}{2}\right)\frac{e^{2iuh}}{s^2+\gamma^2(u)}du]ds\\
&+\frac{i\mu_0 e^{\beta x}}{2\pi^2}\sum_{n=0}^{\infty}a_nQ_n\int_0^{\infty}\frac{s^2e^{\gamma x}}{\gamma(s)(e^{2\gamma h}-e^{-2\gamma h})}[e^{-2\gamma h}\int_{-\infty}^{\infty}G(u)J_{n+1}\left(u\frac{b-a}{2}\right)\frac{1}{s^2+\gamma^2(u)}du\\
&-\int_{-\infty}^{\infty}G(u)J_{n+1}\left(u\frac{b-a}{2}\right)\frac{e^{2iuh}}{s^2+\gamma^2(u)}du]ds,\qquad a\le x\le b.
\end{aligned}\tag{35}$$

So the singular part of stress field can be expressed as follows:

$$\begin{aligned}
\sigma&=-\frac{\mu_0 e^{\beta x}}{4\pi}\sum_{n=0}^{\infty}a_nQ_n\left[\int_0^{\infty}J_{n+1}\left(s\frac{b-a}{2}\right)e^{is(x-\frac{a+b}{2})}ds-\int_{-\infty}^{0}J_{n+1}\left(s\frac{b-a}{2}\right)e^{is(x-\frac{a+b}{2})}ds\right]\\
&=-\frac{\mu_0 e^{\beta x}}{2\pi}\sum_{n=0}^{\infty}a_nQ_nH_n(a,b,x),
\end{aligned}\tag{36}$$

where

$$H_n(a,b,x)=\begin{cases}-(-1)^{\frac{n}{2}}F_1(a,b,x,n), & n=0,2,4,6,\cdots\\ -i(-1)^{\frac{n+1}{2}}F_1(a,b,x,n), & n=1,3,5,7,\cdots\end{cases}\quad(0\le x<a),\tag{37}$$

$$H_n(a,b,x)=\begin{cases}-(-1)^{\frac{n}{2}}F_2(a,b,x,n), & n=0,2,4,6,\cdots\\ i(-1)^{\frac{n+1}{2}}F_2(a,b,x,n), & n=1,3,5,7,\cdots\end{cases}\quad(b<x<2h),\tag{38}$$

$$F_1(a,b,x,n)=\frac{2(b-a)^{n+1}}{\sqrt{(a+b-2x)^2-(b-a)^2}[a+b-2x+\sqrt{(a+b-2x)^2-(b-a)^2}]^{n+1}},\tag{39}$$

$$F_2(a,b,x,n)=\frac{2(b-a)^{n+1}}{\sqrt{(2x-a-b)^2-(b-a)^2}[2x-a-b+\sqrt{(2x-a-b)^2-(b-a)^2}]^{n+1}}.\tag{40}$$

At the left tip of the crack, we obtain the stress intensity factor K_L as

$$\begin{cases}K_L=\lim\limits_{x\to a^-}\sqrt{2(a-x)}\cdot\sigma=\dfrac{\mu_0 e^{\beta a}}{\pi}\sqrt{\dfrac{1}{2(b-a)}}\sum\limits_{n=0}^{\infty}a_nQ_nR_n,\\ R_n=\begin{cases}(-1)^{\frac{n}{2}}, & n=0,2,4,6,\cdots,\\ i(-1)^{\frac{n+1}{2}}, & n=1,3,5,7,\cdots.\end{cases}\end{cases}\tag{41}$$

At the right tip of the crack, we obtain the stress intensity factor $K_{\rm R}$ as

$$K_{\rm R} = \lim_{x\to b^+}\sqrt{2(x-b)}\cdot\sigma = \frac{\mu_0 {\rm e}^{\beta b}}{\pi}\sqrt{\frac{1}{2(b-a)}}\sum_{n=0}^{\infty} a_n Q_n R'_n,$$
$$R'_n = \begin{cases} (-1)^{\frac{n}{2}}, & n=0,2,4,6,\cdots, \\ -{\rm i}(-1)^{\frac{n+1}{2}}, & n=1,3,5,7,\cdots. \end{cases} \tag{42}$$

The dimensionless stress intensity factors $K_{\rm L}$ and $K_{\rm R}$ are calculated numerically. From Refs.[14–16], it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of the infinite series to Eq.(34) are obtained. The crack surface loading $-\tau_0(x)$ will simply be assumed to be a polynomial of the form as follows:

$$-\tau_0(x) = -p_0 - p_1 x - p_2 x^2 - p_3 x^3. \tag{43}$$

Since the problem is linear, the results can be superimposed in any suitable manner. The results are obtained by taking only one of the four input parameters p_0, p_1, p_2 and p_3 nonzero at a time. The normalized non-homogeneity constant β is varied from -2.0 to 2.0, which covers most of the practical cases. The results of present paper are shown in Figs.2–7. From the results, the following observations are very significant:

(i) The aim of this paper is just to give an approach to solve the dynamic fracture problem in the functionally graded materials subjected to the harmonic anti-plane shear stress waves. In this paper, the unknown variables of dual integral equations are the displacement jumps across the crack surfaces,not the dislocation. From the results, it can be obtained that the stress intensity factors are dependent on the crack length, the width of strip, the material parameters and the circular frequency of the incident wave.

(ii) The stress intensity factors increase with the increase in the circular frequency of the incident wave until reaching a peak at $\omega/c \approx 0.75$. Then they come to decrease as shown in Fig.2.

(iii) The stress intensity factors at the left crack tip are smaller than ones at the right crack tip for $\beta > 0$ and $\tau_0(x) = p_0$ as shown in Fig.3. The stress intensity factors at the left crack tip are smaller than ones at the right crack tip for $\beta > 0.25$ with $\tau_0(x) = p_1 x$ and $\beta > 0.6$ with $\tau_0(x) = p_3 x^3$, as shown in Figs.6–7, respectively. However, the stress intensity factors at the left crack tip are larger than ones at the right crack tip for $\beta < 0$ and $\tau_0(x) = p_0$ as shown in Fig.3. The stress intensity factors at the left crack tip are larger than ones at the right crack tip for $\beta < 0.25$ with $\tau_0(x) = p_1 x$ and $\beta < 0.6$ with $\tau_0(x) = p_3 x^3$, as shown in Figs.6–7, respectively.

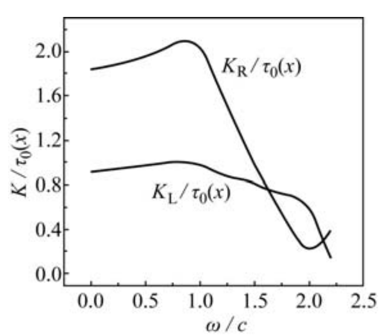

Fig.2 The stress intensity factor *versus* ω/c_1 for $h = 1.5, a = 0.1, b = 2.0, \beta = 0.2$ and $\tau_0(x) = p_0$

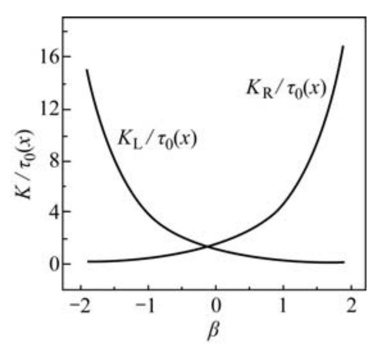

Fig.3 The stress intensity factor *versus* β for $h = 1.5$, $a = 0.1$, $b = 2.0$, $\omega/c_1 = 0.4$ and $\tau_0(x) = p_0$

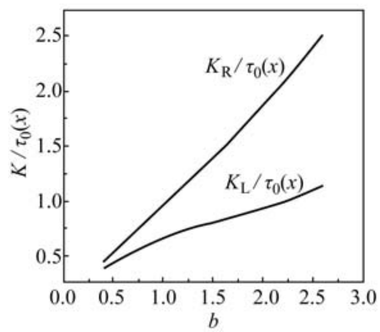

Fig.4 The stress intensity factor *versus* b for $h = 1.5, a = 0.1$, $\omega/c_1 = 0.2$, $\beta = 0.2$ and $\tau_0(x) = p_0$

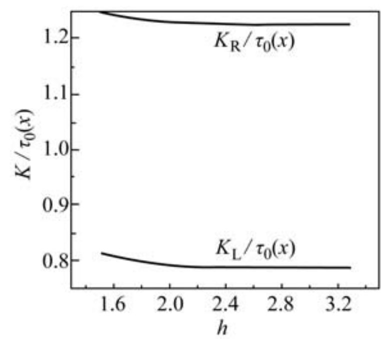

Fig.5 The stress intensity factor *versus* h for $a = 0.1, b = 2.0$, $\omega/c_1 = 0.2$, $\beta = 0.2$ and $\tau_0(x) = p_0$

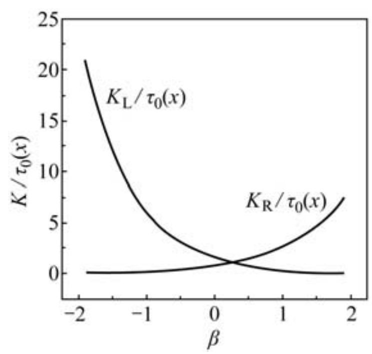

Fig.6 The stress intensity factor *versus* β for $a = 0.4, b = 2.0$, $\omega/c_1 = 0.2$, $h = 1.5$ and $\tau_0(x) = p_1 x$

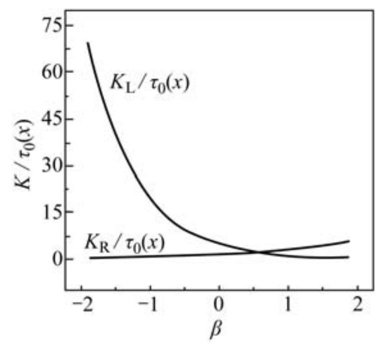

Fig.7 The stress intensity factor *versus* β for $a = 0.4, b = 2.0$, $\omega/c_1 = 0.2$, $h = 1.5$ and $\tau_0(x) = p_3 x^2$

(iv) As shown in Fig.4, the stress intensity factors increase with the increase of crack length. As shown in Fig.5, it can be obtained that the variation of the strip width has the rather insignificant influence on the stress intensity factors.

(v) The stress intensity factors at the right crack tip increase with the increase of non-homogeneity constant β. However, the stress intensity factors at the left crack tip decrease with the increase of non-homogeneity constant β as shown in Figs.3, 6 and 7.

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